2D Element

CE 530624

Constant Strain Triangle (CST), or Linear Triangle

Displacement field (linear in x and y)

\[ u = \beta_1 + \beta_2 x + \beta_3 y \]
\[ v = \beta_4 + \beta_5 x + \beta_6 y \]

Strain field (constant)

\[ \varepsilon_x = \frac{\partial u}{\partial x} = \beta_2 \]
\[ \varepsilon_y = \frac{\partial v}{\partial y} = \beta_6 \]
\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \beta_3 + \beta_5 \]
Constant Strain Triangle (CST), or Linear Triangle

Constants $\beta_1$-$\beta_3$ are obtained from:

\[
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix} =
\begin{bmatrix}
1 & x_1 & y_1 \\
1 & x_2 & y_2 \\
1 & x_3 & y_3
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3
\end{bmatrix}
\]

Constants $\beta_4$-$\beta_6$ are obtained from:

\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
1 & x_1 & y_1 \\
1 & x_2 & y_2 \\
1 & x_3 & y_3
\end{bmatrix}
\begin{bmatrix}
\beta_4 \\
\beta_5 \\
\beta_6
\end{bmatrix}
\]

Tedious algebra is needed to get to the form: $\varepsilon = B d$

Using principle of virtual work:

\[ k = B^T E B \ell A \]

where $\ell$ is element thickness, and $A$ is element area

constitutive matrix $E$ depends on the problem (plane stress or plane strain)

- Early, simple element
- Does not work very well, especially in pure bending
  - in pure bending strains are linear, not constant
- Will converge (slowly) to correct results if mesh is repeatedly refined

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Linear Strain Triangle (LST), or Quadratic Triangle

Displacement field (quadratic in $x$ and $y$)

\[
\begin{align*}
u &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 \\
v &= \beta_7 + \beta_8 x + \beta_9 y + \beta_{10} x^2 + \beta_{11} xy + \beta_{12} y^2
\end{align*}
\]

Strain field (linear)

\[
\begin{align*}
\varepsilon_x &= \beta_2 + 2 \beta_4 x + \beta_5 y \\
\varepsilon_y &= \beta_9 + \beta_{11} x + 2 \beta_{12} y \\
\gamma_{xy} &= (\beta_5 + \beta_6) + (\beta_5 + 2 \beta_{10}) x + (2 \beta_6 + \beta_{11}) y
\end{align*}
\]

- Works much better than constant strain triangle
  - exact results for pure bending because of the linear strain filed
- Possible to have initially curved sides (isoparametric formulation)
- Requires numerical integration of $k = \int B^T E B dV$ if sides are initially curved
Element Performance - Triangles

Analytical Solution

\[ \sigma_{xB} = \frac{Mc}{I} = 300 \]
\[ \delta_c = \frac{PL^3}{3EI} + \frac{6PL}{5AG} = 1.031 \]

Unit thickness, \( v = 0.3 \)

2-node Beam Element

\[ \sigma_{xB} = 300 \]
\[ \delta_c = 1.031 \]

Linear Strain Triangle (LST)

\[ \sigma_{xB} = 254 \]
\[ \delta_c = 0.987 \]

Constant Strain Triangle (CST)

\[ \sigma_{xB} = 71 \]
\[ \delta_c = 0.264 \]
\[ \sigma_{xB} = 250 \]
\[ \delta_c = 0.924 \]


Bilinear Quadrilateral (Q4)

Displacement field

\[ u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy \]
\[ v = \beta_5 + \beta_6 x + \beta_7 y + \beta_8 xy \]

\( u \) and \( v \) are products of two linear polynomials \( \rightarrow \) bilinear

Strain field

\[ \varepsilon_x = \beta_2 + \beta_4 y \]
\[ \varepsilon_y = \beta_7 + \beta_8 x \]
\[ \gamma_{xy} = (\beta_3 + \beta_6) + \beta_4 x + \beta_8 y \]

4 vertex nodes
2 displacements per node
8 nodal displacements

- In most problems works better than the constant strain triangle
- Functions well under direct or shear loads
- Fully integrated Q4 element performs poor under pure bending
- Converges properly with mesh refinement.
Bilinear Quadrilateral (Q4) – Shear stresses

Element cannot model pure bending. The displacement along the sides is straight.

- Shear stresses developed as a result of the straight top and bottom sides
- Results in spurious, or parasitic shear under pure bending. This spurious shear absorbs strain energy.

Required value of $M_2$ to form same angle $\theta$ in both case of pure bending and the bilinear quadrilateral:

$$M_2 = \frac{1}{1+\nu} \left[ \frac{1}{1-\nu} + \frac{1}{2} \left( \frac{a}{b} \right)^2 \right] M_1$$

- As $a/b$ increases without limit, so does $M_2$
- Element becomes infinitely stiff in bending
- This is called “locking”

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>$M_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.47$M_1$</td>
</tr>
<tr>
<td>2</td>
<td>2.67$M_1$</td>
</tr>
<tr>
<td>4</td>
<td>7.47$M_1$</td>
</tr>
</tbody>
</table>

$\nu = 0.25$

Improved Bilinear Quadrilateral (Q6)

Two internal degrees-of-freedom are added, which are essentially pure bending degrees-of-freedom.

- Internal degrees-of-freedom are not connected to adjacent elements
- Boundary displacements can be incompatible

- Element is said to be nonconforming
- Can represent pure bending exactly only if the element is rectangular
- Element does properly converge

ADINA note: This element can be selected by choosing incompatible modes for the 4-node element in the “Element Group” command.
**Quadratic Quadrilateral – 8 node element (Q8)**

- 4 vertex & 4 midside nodes
- 2 displacements per node
- 16 nodal displacements

Element can exactly represent pure bending if it is rectangular.

**Displacement field**

\[
\begin{align*}
u &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 + \beta_7 x^2 y + \beta_8 xy^2 \\
v &= \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} x^2 + \beta_{13} xy + \beta_{14} y^2 + \beta_{15} x^2 y + \beta_{16} xy^2 
\end{align*}
\]

**Strain field**

\[
\begin{align*}
\varepsilon_x &= \beta_2 + 2 \beta_4 x + \beta_6 y + 2 \beta_7 xy + \beta_8 y^2 \\
\varepsilon_y &= \beta_{11} + \beta_{13} x + 2 \beta_{14} y + \beta_{15} x^2 + 2 \beta_{16} xy \\
\gamma_{xy} &= (\beta_3 + \beta_{10}) + (\beta_5 + 2 \beta_{12}) xy + (2 \beta_6 + \beta_{13}) y + \beta_7 x^2 + \beta_{16} y^2 + 2(\beta_8 + \beta_{15}) xy
\end{align*}
\]

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**Biquadratic Quadrilateral – 9 node element (Q9)**

- 4 vertex, 4 midside & 1 center node
- 2 displacements per node
- 18 nodal displacements

\(u\) and \(v\) are products of two quadratic polynomials → biquadratic

**Displacement field**

\[
\begin{align*}
u &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 + \beta_7 x^2 y + \beta_8 xy^2 + \beta_9 x^2 y^2 \\
v &= \beta_{10} + \beta_{11} x + \beta_{12} y + \beta_{13} x^2 + \beta_{14} xy + \beta_{15} y^2 + \beta_{16} x^2 y + \beta_{17} xy^2 + \beta_{18} x^2 y^2 
\end{align*}
\]

Element subjected to angular distortion

9-node element: can exactly represent a quadratic displacement field
8-node element: can only represent a linear displacement field exactly

**Element subjected to curved edge distortion**

Both 8-node and 9-node elements can only represent a linear displacement field exactly.
Isoparametric Elements

Natural coordinate system is used.
Element sides are bisected by the natural coordinate axes

\[
\begin{align*}
    x &= \sum N_i x_i \\
    y &= \sum N_i y_i \\
    u &= \sum N_i u_i \\
    v &= \sum N_i v_i
\end{align*}
\]

\[
\begin{align*}
    N_1 &= \frac{1}{4}(1 - \xi)(1 - \eta) \\
    N_2 &= \frac{1}{4}(1 + \xi)(1 - \eta) \\
    N_3 &= \frac{1}{4}(1 + \xi)(1 + \eta) \\
    N_4 &= \frac{1}{4}(1 - \xi)(1 + \eta)
\end{align*}
\]

Isoparametric bilinear quadrilateral (Q4)
- Removes the restriction to rectangular shape
- Shear locking defect remains – addressed by the isoparametric Q6 element

Other 2D isoparametric elements: LST, Q8 & Q9
- Possibly to have initially curved sides (due to the midside nodes)
- These geometric distortions are usually detrimental to accuracy
- Very useful for modeling arbitrary geometry, holes or fillets
- Behave much like their corresponding (nonisoparametric) predecessors

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Element Performance - Quadrilaterals

Analytical Solution
\[
\begin{align*}
    \sigma_{xb} &= Mc/I = 300 \\
    \delta_c &= PL^3/3EI + 6PL/5AG = 1.031
\end{align*}
\]

Unit thickness, \( v = 0.3 \)

2-node Beam Element
\[
\begin{align*}
    \sigma_{xb} &= 300 \\
    \delta_c &= 1.031
\end{align*}
\]

Bilinear Quadrilateral (Q4)
\[
\begin{align*}
    \sigma_{xb} &= 200 \\
    \delta_c &= 0.693
\end{align*}
\]

Improved Bilinear Quad. (Q6)
\[
\begin{align*}
    \sigma_{xb} &= 270 \\
    \delta_c &= 1.016
\end{align*}
\]

Quadratic Quadrilateral (Q8)
\[
\begin{align*}
    \sigma_{xb} &= 300 \\
    \delta_c &= 1.028
\end{align*}
\]

\[
\begin{align*}
    \sigma_{xb} &= 190 \\
    \delta_c &= 0.502
\end{align*}
\]

\[
\begin{align*}
    \sigma_{xb} &= 281 \\
    \delta_c &= 0.980
\end{align*}
\]

\[
\begin{align*}
    \sigma_{xb} &= 278 \\
    \delta_c &= 1.035
\end{align*}
\]

Element Performance – Quadratic Quadrilaterals Q8 & Q9

Analytical Solution
\[
\begin{align*}
\sigma_x &= \frac{Mc}{I} = 60 \\
\sigma_y &= 0 \\
\tau_{xy} &= 0
\end{align*}
\]

Eight-node quadrilateral (Q8)
Stress distribution same as analytical solution
\[
\begin{align*}
\sigma_x &= 4.0 \\
\sigma_y &= 22.4 \\
\tau_{xy} &= 2.0
\end{align*}
\]
\[
\begin{align*}
\sigma_x &= 10.0 \\
\sigma_y &= -13.6 \\
\tau_{xy} &= -6.1
\end{align*}
\]

Nine-node quadrilateral (Q9)
Stress distribution same as analytical solution
\[
\begin{align*}
\sigma_x &= 41.8 \\
\sigma_y &= 38.3 \\
\tau_{xy} &= 21.8
\end{align*}
\]
\[
\begin{align*}
\sigma_x &= 28.5 \\
\sigma_y &= 46.8 \\
\tau_{xy} &= -7.2
\end{align*}
\]


Solid Elements
Elements similar to the 2D elements can be developed for three-dimensional solids.
Solid elements have similar behavior to the corresponding 2D element.

<table>
<thead>
<tr>
<th>2D element</th>
<th>Corresponding 3D element</th>
</tr>
</thead>
<tbody>
<tr>
<td>bilinear quadrilateral (Q4)</td>
<td>8 noded brick</td>
</tr>
<tr>
<td>8-node quadrilateral (Q8)</td>
<td>20-node solid element</td>
</tr>
<tr>
<td>9-node quadrilateral (Q9)</td>
<td>27-node solid element</td>
</tr>
</tbody>
</table>
Element shapes and transitions

Poorly shaped elements
- Large aspect ratio
- Near triangle
- Highly skewed
- Off-center node
- Strongly curved side

Element transitions
- Poor
- Improved

Element transition in two-dimensions

Work-Equivalent Loads – Surface Traction

Surface traction loads and body forces cannot be applied directly to a FE model. They must be converted to work-equivalent (consistent) nodal loads.

- Nodal loads that do the same amount of work as the loads they are replacing.

**Work-equivalent loads for surface traction**

- Linear varying load on linear displacement edge

\[
\begin{bmatrix}
F_A \\
F_B
\end{bmatrix} = \frac{L}{6} \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
q_A \\
q_B
\end{bmatrix}
\]

- Quadratic varying load on quadratic displacement edge

\[
\begin{bmatrix}
F_A \\
F_B \\
F_C
\end{bmatrix} = \frac{L}{30} \begin{bmatrix}
4 & 2 & -1 \\
2 & 16 & 2 \\
-1 & 2 & 4
\end{bmatrix} \begin{bmatrix}
q_A \\
q_B \\
q_C
\end{bmatrix}
\]

- Uniform load \( \rightarrow 1/6 : 2/3 : 1/6 \) or \( 1 : 4 : 1 \)
- Parabolic load \( \rightarrow 1/10 : 4/5 : 1/10 \) or \( 1 : 8 : 1 \)
**Work-Equivalent Loads – Body Forces**

Work-equivalent loads associated with element weight

- **Constant strain triangle CST**
  - $W/3$
  - $W/3$
  - $W/3$

- **Bilinear quadrilateral Q4 & Q6**
  - $W/4$
  - $W/4$

- **Linear strain triangle LST**
  - $W/3$
  - $W/3$
  - $W/3$

- **Quadratic quadrilateral Q8**
  - $W/12$
  - $W/3$
  - $W/12$
  - $W/3$

---

**Application of Work-Equivalent Load Concept to Supports**

- **Cantilever beam with bilinear quadrilaterals**

- **Cantilever beam with quadratic quadrilaterals**

- **Bilinear quadrilaterals on elastic foundation**
  - $1k$
  - $2k$
  - $1k$

- **Quadratic quadrilaterals on elastic foundation**
  - $1k$
  - $4k$
  - $2k$
  - $2k$

*Boundary conditions are VERY OFTEN misrepresented. Be careful with them.*

Cook, 1995
Loads: Application

Analyze a cantilever beam under a concentrated load at the end using 2D elements.

![Parabolic Shear Stress](image)

Linear problems: Loads maintain original orientation in space.
Followers (nonlinear): direction changes as structure deforms.
Example: pressure on a membrane.

Dead loads: applied instantaneously to structure (gravity is turned on).
Examples: footings cast on soil, or tall building construction.

Boundary Conditions

Saint Venant's Principle
- If different approximations to boundary conditions are statically equivalent
- Resulting solutions are valid at regions sufficiently far away from support.
**Determination of Stresses**

- Nodal displacements, $d$ – primary unknowns in the FE method
- Stresses, $\sigma$ – derived quantities $\sigma = E\varepsilon + \sigma_0 = EBd + \sigma_0 = E\Delta N d + \sigma_0$
  - Above equation is applied at element level, not globally
- Stresses can theoretically be determined at any point in the element
- Generally, stresses are most accurate at the integration points
- Often interested in stresses at nodes, or surfaces (obtained by extrapolation)
- Element by element stress field may be discontinuous
  - Stress at same node from different elements will vary
  - Can average stress at nodes to obtain a “smooth” stress field
  - Differences in stresses at a node, however, is a good indication of the accuracy of the solution
  - Thus, be careful, when averaging stresses
  - Some problems, by their very nature, will have stress discontinuities at the nodes (e.g., parts joined by a shrink fit; two different materials joined together, two parts with different thicknesses joined together).
- Stresses can be reported in local or global coordinates
  - Beam: local coordinates are useful since flexural stress is a normal stress in the axial direction of the beam
  - Plate with hole: global coordinates are useful since interested in vertical and horizontal stress, irrespective of local direction

**Element Connections – 2D Elements**

Elements of different types can be connected to one another, but not in completely arbitrary fashion.

**How not to connect 2D elements**

- **Cook (1995)**

  - **A** not a connection: CST has no side node.
  - **B-C** a two-node edge (CST, Q4 or Q6) and a three-node edge (LST, Q8 or Q9)
    - side node is left unconnected
    - mismatch of displaced shapes (straight line vs. parabola)
  - **C-D** two two-node edges (CST, Q4 or Q6) and a three-node edge (LST, Q8 or Q9)
    - mismatch of displaced shapes (two straight lines and a parabola)
  - **E-F** same as C-D
  - **G-H** three-node edges connected such that side nodes are joined to corner nodes
    - mismatch of displaced shapes (two different parabolas)

Such connections are not fatal. They cause poor results locally, but the effect dies away with distance (Saint-Venant’s Principle)
Element Connections – Beams and 2D Elements

Coupled shear wall structure

Ad-hoc solution:
transfers moment, but
function of beam stiffness

Add constraint equation

\[ \theta_{21} = \frac{u_1 - u_3}{b} \quad \text{or} \quad \theta_{21} = \frac{v_1 - v_3}{a} \]

Be careful with signs.

Nature of Finite Element Solution

- Finite element method is a form of Rayleigh-Ritz method.
  Classical approximation technique originated in 1870 and generalized in 1909
- Rayleigh-Ritz method
  - Assume a displacement field (e.g. linear, quadratic, sine…) over the entire
    body in terms of generalized coordinates \( \beta_i \) (no elements, and no nodes)
  - Displacement field requirements:
    - satisfy compatibility conditions within the body, and
    - satisfy geometric (displacement) boundary conditions
  - Form an energy expression that includes:
    - strain energy of the body
    - work done by applied loads
  - Minimization of the total energy with respect to \( \beta_i \) yields simultaneous
    algebraic equations that can be solved for the unknown \( \beta_i \)
  - Obtained solution is a “lower bound” solution; displacements are either exact
    or too small as compared to the exact solution of the mathematical model.
  - Assuming displacement field has in effect applied additional constraints to
    the problem. They prevent the exact displacement field from appearing.
Nature of Finite Element Solution, Continued

- Finite element method specifics
  - The assumed displacement field applies to a single element.
    The displacement field of the entire body is obtained in piecewise fashion from individual element displacement fields.
  - The primary unknowns are nodal displacements instead of generalized coordinates $\beta_i$.
- Finite element solution will also yield a “lower bound” solution if
  - work-equivalent nodal loads are used
  - elements are compatible
  - elements are integrated exactly
- “Lower bound” solution
  - Does not mean that all displacements are too small
  - Work done by applied loads is too small
  - Displacements are too small in an average sense
  - If load is a single force or moment, the corresponding displacement at the loaded point is either exact or too small

Choice of Numerical Integration Order

- Full integration
  - Order of integration which gives the exact stiffness matrix for a geometrically undistorted element
  - Will usually give acceptably small errors for geometrically distorted elements
  - Gauss quadrature:
    - 2x2 for bilinear quadrilateral (Q4)
    - 3x3 for quadratic quadrilateral (Q8)
- Reduced integration
  - Level less than full integration
  - Usually results in a more “flexible” stiffness matrix, since it is not including higher order terms of the stiffness matrix
  - May improve overall solution since by its nature, the finite element method is too stiff
  - Can result in instabilities if too few Gauss points are being used
  - Some software packages have stabilization schemes

See Cook, Fig. 4.6-1 and 4.6-2 for illustrations of instabilities
Element Performance: Order of Numerical Integration

<table>
<thead>
<tr>
<th>Element Type</th>
<th>Integration Order</th>
<th>Ratio of Finite Element Value to Analytical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Rectangular Elements</td>
</tr>
<tr>
<td>8 node</td>
<td>2 x 2</td>
<td>$\sigma_{xB}$</td>
</tr>
<tr>
<td>8 node</td>
<td>3 x 3</td>
<td>1.129</td>
</tr>
<tr>
<td>9 node</td>
<td>2 x 2</td>
<td>1.000</td>
</tr>
<tr>
<td>9 node</td>
<td>3 x 3</td>
<td>1.141</td>
</tr>
</tbody>
</table>


Mesh Refinement

- **h-refinement**
  - $h$ refers to a measure of element size
  - The element size is decreased
  - If the new mesh contains the old mesh, the convergence will be monotonic

- **p-refinement**
  - $p$ refers to degree of the highest complete polynomial in the element displacement field
  - Element type is changed without changing element size
  - Provides rapid convergence, but typically an element type is chosen and used throughout the analysis

- **r-refinement**
  - $r$ refers to rearrange
  - Number of elements and type of elements are not changed, but the elements are rearranged
  - Convergence may not be monotonic
**Substructuring**

Analyzing a large finite element model as a collection of component FE models. Substructure is sometimes called a "superelement."

**Nodes:**
- **Condensed node:** not connected to any node in main structure
- **Retained node:** connected to at least one node in main structure

**Advantages of substructuring:**
1. Repetitive structure. Only need to model part once. Saves solution time and input time.
2. Nonlinearities confined to a single part of the structure. Model elastic part as a substructure.
3. Little interaction between different parts. Individual substructure can be analyzed and designed separately from entire structure, and only occasionally does entire structure need to be solved.