Beams and Mats on Elastic Foundation

CE 530624

Soil-structure interaction (SSI)

Distribution of bending moments and shear forces in a beam, or a mat on elastic foundation depends on:
- nature of loading (known)
- distribution of contact (soil) pressure (unknown, but satisfies equilibrium)

Distribution of contact pressure in the elastic continuum depends on:
- nature of loading (known)
- soil properties: elastic modulus, $E$, and Poisson’s ratio, $\nu$ (uncertain)
- beam/mat properties: elastic modulus, $E$, moment of inertia of a beam, $I$, and thickness, $h$, and Poisson’s ratio, $\nu$, for a mat. (unknown geometry)

The effects are coupled, hence the phenomena is termed soil-structure interaction (SSI).

Soil-structure interaction definition

Soil-structure interaction

- The deformation, strain and stress along the structure depend upon the distribution of deformation, strain and stress in the soil.
- However, the distribution of deformation, strain, and stress in the soil depend upon the deformation, strain and stress along the structure.
**Distribution of contact pressure**

Look at two cases at the opposite end of the spectrum:
- Rigid structure on elastic continuum (example: rigid punch, die)
  - uniform settlement
  - nonuniform contact pressure (higher at edges)
- Flexible structure on elastic continuum (example: auto tire, water bed)
  - uniform contact pressure
  - nonuniform settlement

In both cases the ratio of unit contact pressure at a point and settlement at that point is not constant along the structure.

A useful approximation is to assume that the contact pressure at every point is proportional to the settlement (deflection) occurring at that point and is independent of pressures or deflections produced elsewhere.
- Winkler model

**Winkler model of a beam on elastic foundation**

Introduced by E. Winkler in 1867 and implemented for the first time in the analysis of railroad track by H. Zimmermann in 1888.

Physically, Winkler’s idealization of the soil medium consist of an infinite system of elastic springs that act independently of one another.

![Diagram of Winkler model](image)

\[ p = k w \]

Units: (L)

\[ S = k \frac{p}{w} \]

Units: (L)

\[ k = \frac{p}{w} \]

The soil stiffness, \( k \), can be obtained by multiplying the modulus of subgrade reaction, \( k_s \), with the beam width, \( b \).

\[ k = k_s b \]

\[ p = k w = k_s b w \]

The modulus of subgrade reaction, \( k_s \), is the slope of the soil pressure versus deflection relationship from a plate load test (PLT); units: (L)

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**Classical solution of a beam on elastic foundation**

FBD

![](image)

\[ \sum Y = 0 \Rightarrow -Q + (Q + dQ) - kw\frac{dQ}{dx} + q dx = 0 \Rightarrow \frac{dQ}{dx} = kw - q \]

Recall:

\[ Q = \frac{dM}{dx} \]

\[ \frac{dQ}{dx} = \frac{d^2 M}{dx^2} = kw - q \]

Recall:

\[ EI \frac{d^2 w}{dx^2} = -M \Rightarrow EI \frac{d^4 w}{dx^4} = \frac{d^2 M}{dx^2} = -kw + q \]

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**Classical solution, continued**

Governing differential equation for the deflection of a prismatic beam supported on an elastic foundation:

\[ w \] — deflection (settlement) (L)

\[ E I \frac{d^4 w}{dx^4} + kw = q \]

\[ q \] — applied distributed load (F/L)

\[ k \] — soil (foundation) stiffness (F/L²)

General solution:

\[ E I \frac{d^4 w}{dx^4} + kw = q \]

\[ w = e^{\lambda x} (C_1 \cos \lambda x + C_2 \sin \lambda x) + e^{-\lambda x} (C_3 \cos \lambda x + C_4 \sin \lambda x) \]

where: \( C_1, \ldots, C_4 \) are determined from BC’s at the end of the beam, and

\[ \lambda = \frac{1}{\sqrt{4EI}} \]

Characteristic Length is defined as a reciprocal of \( \lambda \) (CL=1/A)

- it includes:
  - flexural rigidity (stiffness) of the beam, \( EI \)
  - stiffness of the elastic continuum, \( k \)
- it influences the shape of the beam (deformed shape)
- it is a measure of the soil-structure interaction
Classical solution, continued

Look at two cases at the opposite end of the spectrum:

- $\lambda L$ is small – “short” or “rigid” beam: $\lambda L < \pi/4$ (Hetenyi), some say $\lambda L < \pi$
  - beam is very stiff with respect to soil
  - applied load will cause deflections of beam and soil springs to considerable distance from point of action of load
  - treat as a finite beam

- $\lambda L$ is large – “long” or “flexible” beam: $\lambda L > \pi$ (Hetenyi)
  - beam is flexible with respect to soil
  - applied load has only local effect on deflections and stresses
    - bending is localized, forces applied at one end of beam have negligible effect on other end (or at a distance $L_y > \pi L$ to $1.5\pi L$)
    - vertical loads (of approx equal magnitude) applied at distance $L_y > 1.5\pi L$ do not interact, or
    - nature of support at $L_y > 1.5\pi L$ from load has no effect
  - treat as an infinite beam

Classical solution literature

- Hetenyi M. (1946) Beams on Elastic Foundation (out of print)
  Specific cases considered:
  - finite beam, central point load
  - finite beam, arbitrary point load
  - finite beam, two symmetrically placed point loads
  - finite beam, symmetrically placed uniform load
  - cantilever beam, point load at end
  - cantilever beam, uniformly distributed load

- Roark R.J. and Young, W.C. (1975) Formulas for Stress and Strain
  in Roark and Young tables $\beta$ corresponds to $\lambda$
  
  $\beta = \frac{k_y b_y}{4EI}$

  - $k_y = k_z$ – modulus of subgrade reaction (F/L²)
  - $b_y$ – beam width (L)

  - continuous beams and mats
  - slabs and plates (3 dimensional problem)

Limitations of the Winkler model

Limitations:

- Does not provide continuous settlement of ground surface beyond end of beam under “heavy loads”.

- Does not provide separation between soil and structure under “light loads”. Springs carry tension.

- Requires that end springs have same stiffness as interior ones.

- Cannot account for rigid support (rock) below shallow elastic layer.

Limitations of the Winkler model, continued

Limitations:

- Cannot account for settlement and participation of soil beyond the edge of the beam.

  Load applied to an actual beam on elastic half-space results in deformation both under load and outside load.

- Soil is represented by just one parameter, the soil stiffness, $k$.
  Techniques are available to treat the soil as continuous, linear elastic, isotropic, half-space completely defined in terms of two parameters – modulus of elasticity, $E_s$, and Poisson’s ratio, $\nu_s$.
  - Therefore cannot expect simultaneous correspondence among all derived quantities (displacement, slopes, moments, etc.) between the Winkler model and the elastic half-space model.
  - For a particular $E_s$ and $\nu_s$ there exists a particular $k$ that will yield the same maximum displacement.
  - However, a different $k$ may be required to yield the same maximum moment in the beam.
Correlation between $k$ and elastic constants $E$, and $v_s$:

Vesic’s relation (1961)

$$k = \frac{0.65E_s}{E}$$

- $E_s$ – Young’s modulus of the soil (Pa/L²)
- $v_s$ – Poisson’s ratio of the soil
- $E$ – beam flexural stiffness (Pa/L²)
- $b$ – beam width (L)

- results are in close agreement (within 10%) with half-space solution for all variables
- since the 12th root is near unity, it is often neglected with little effect

For beams with typical ranges of $EI$, the maximum moments obtained from linear elastic half-space solutions and beam on elastic foundation solution are identical if:

$$k = \frac{0.95E_s}{1-v_s^2}$$

Modulus of subgrade reaction

The modulus of subgrade reaction, $k_s$, is defined as the slope of the soil pressure versus deflection relationship from a plate load test (PLT).

Plate load test:
- Apply force, $P$, to “rigid” loading plate with contact area, $A$
- Measure deflection, $\delta$, as a function of applied force load, $P$, or the assumed contact pressure, $q(x,y) = P/A = \text{constant}$

- Resulting load (pressure) vs. deflection curve is nonlinear
  - Choose initial or tangent value for small loads
  - Choose secant value for larger loads

Issues with the plate load test

- Plate load test is rarely used in practice any more
- Plate is assumed as being rigid
- Pressure distribution is assumed as being uniform
  - actual pressure distribution is unknown, but it is certain that the stresses are higher at the edges of the (rigid) loading plate.
- Uniform displacements yield average modulus of subgrade reaction, $k_s$
- Test is subject to size effects
  - must keep plate small to limit load to a reasonable value
  - yet large enough to load a significant volume, or depth of soil
- Test only measures the response of near surface soils

Typical values for physical constants

- Modulus of subgrade reaction, $k_s$
  - Table 9-1 from Bowles (1988)
  - Figure from NAVFAC DM-7.2
- Young’s modulus, $E_s$ (Lambe and Whitman, 1969)
  - sand: Table 12.4
  - clay: Figure 5.10 (function of undrained shear strength, $c_u$, and plasticity index, PI)
- Poisson’s ratio, $v_s$
  - sand: $v_s = 0.2$ to $0.3$
  - undrained clay: $v_s = 0.5$
Solution sensitivity to the soil stiffness value

The Winkler model requires in its formulation a single soil parameter, the soil stiffness, \( k \).

Soil stiffness is obtained in one of two ways:
- from the modulus of subgrade reaction: \( k = k_b b \)
- from correlations with elastic constants \( E_s \) and \( v_s \) \( E_s = \frac{0.65E_f}{1 - v_s} \).

Uncertainty associated with the estimation of soil properties can be substantial, but:
- For given \( EI \) and \( L \), a significant change in foundation stiffness, \( k \), is required to change \( AL \) from 0.1 (rigid) to 10 (flexible) \( \lambda = \frac{1}{\frac{4EI}{k}} \).
  - Since \( \lambda \) is proportional to \( k^{-1} \), a 1x10^6 increase in \( k \) is required to affect a 1x10^2 change in \( AL \) (to increase \( AL \) from 0.1 to 10)
  - Therefore small changes in \( k \) will not significantly affect behaviour
  - A 20% uncertainty in \( k \) corresponds to a 5% uncertainty in \( AL \).

FEM application of the Winkler model

Winkler model is very suitable for FEM modeling
- mesh beam with multiple beam elements
- at each node use either
  - a "grounded" spring element with stiffness \( k_{spring} \), or
  - a truss element that satisfies \( k_{spring} = E_sA \times L \times t \).

Determination of individual spring stiffness, \( k_{spring} \):

\[
P = p \times L = k \times l \times w = k_{spring} \times w
\]

\[
k_{spring} = k \times l \times w
\]

Units
- \( k_s \) - modulus of subgrade reaction \( (F/L)^2 \)
- \( k \) - soil (foundation) stiffness \( (F/L)^2 \)
- \( l_s \) - tributary length \( (L) \)
- \( k_{spring} \) - spring stiffness \( (F/L) \)

Destination Arrays

Assembly of Global Stiffness Matrix, \( K \)
Mats on elastic foundation

Mat foundation is a large reinforced concrete slab (plate) resting on soil. It ties several column lines together and transfers loads to the supporting soil.

Mat foundation is considered a viable alternative to spread footings when the area under the would be spread footings is more than half the total footprint of the structure.

It's use implies rather large column loads, or very soft soils, or both.

Mat foundation's inherent ease for waterproofing makes it an attractive option in cases where the bottom of the structure is under the water table.

Classical solution of a mat on elastic foundation

Governing differential equation for the out-of-plane deflection, $w$, of a plate on elastic foundation:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} \left[ -k_w + q_z \right]$$

- $q_z = q(x,y)$ – applied pressure
- $k_w$ – modulus of subgrade reaction
- $D = \frac{E h^3}{12(1-v^2)}$ – plate flexural rigidity (stiffness)
- $h$ – mat (plate) thickness
- $w = w(x,y)$ – deflection (settlement)

Closed form solutions are limited to very few combinations of load and boundary conditions:

- rectangular plate with simply supported edges
- rectangular plate with opposing simply supported edges and loading parallel to simply supported edges
- circular plate

Solution methods for mats on elastic foundation

- "Quick and dirty" method
  - Divide mat into several strips and treat each strip as an independent beam on elastic foundation
  - Neglects interaction between strips
  - Overly conservative solution, rarely used in practice

- Analytical solutions based on circular plates (Hetenyi, NAVFAC)
  - Must convert from polar to Cartesian coordinates

- Numerical approaches
  - Plate elements resting on "grounded" spring elements
    (Use the large deflections analysis option if differential deflections approach half mat thickness. In such cases, significant in-plane (membrane) stresses are generated. Plate is much stiffer than small deflection solution indicates.)
  - PCI program MATS

Abaqus model